Title: **Runge-Kutta Discontinuous Galerkin Method Using WENO-Type Limiters: Three-Dimensional Unstructured Meshes.**

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This paper presents a numerical procedure for the solution of the 3-D hyperbolic conservation equation:

$$\begin{aligned} u_t + f(u)_x + g(u)_y + r(u)_z &= 0, \\ u(x,y,z,0) &= u_0(x,y,z) \end{aligned} \tag{1}$$

that achieves high order accuracy and sharp, non-oscillatory transitions. Being those two very important characteristics intensely sought after by computational fluid dynamics (CFD) practitioners since the very early years.

The procedure presented in this paper is based on the Discontinuous Galerkin method and can be used over a 3-D unstructured mesh, making its application to the solution of practical CFD problems straightforward.

Discontinuous Galerkin (DG) methods were introduced in the 1970's for steady state linear hyperbolic equations. Later developments extended the methodology to nonlinear time-dependent hyperbolic conservation laws by combining high order Runge-Kutta time discretizations with DG discretization in space. High order methods need limiters to control the tendency to produce oscillatory solutions when dealing with systems that develop shock fronts or discontinuities in their solution. For that function the authors have developed high order Weighted Essentially Non-Oscillatory (WENO) and Hermitian WENO (HWENO)reconstruction schemes, based on a finite volume 3-D unstructured mesh.

The scheme presented work by:

- identifying the troubled cells under a WENO-type limiting, using a Total Variation Bounded (TVB) minmod-type limiter;

- obtaining the polynomial solution inside the troubled cells by WENO or HWENO reconstructions, using cell averages or derivative averages of neighboring tetrahedrons;
- while retaining the original cell averages of the troubled cells

The paper is clearly laid out, detailed and contains lots of references. The authors present eight numerical examples that show the capabilities and characteristics of the proposed procedure in dealing with problems with well established solutions. These numerical examples show that the method is stable, accurate, and robust in maintaining accuracy.

See also:

1. Guermond, J.L. et al., *An overview of projection methods for incompressible Flows* , Comput. Methods Appl. Mech. Engrg., **195**, pp 6011-6045, (2006)

2. Zhu, J. et al. *Runge-Kutta discontinuous Galerkin method using WENO limiters II: unstructured meshes* available on-line at:
`http://www.dam.brown.edu/scicomp/media/report_files/BrownSC-2007-19.pdf`

3. Zhu, J. et al. *Runge-Kutta discontinuous Galerkin method using new type of WENO limiters on unstructured mesh* available on-line at:
`http://www.dam.brown.edu/scicomp/media/report_files/BrownSC-2012-08.pdf`

4. Shu, C-W. and Qiu, J. *Brief of finite volume WENO method* available on-line at:
`http://lsec.cc.ac.cn/lcfd/DEWENO/weno.pdf`

5. Cockburn, B, Lin, S-Y. and Shu, C-W. *TVB Runge-Kutta Local Projection Discontinuous Galerkin Finite Element Method for Conservation Laws III.*, IMA Preprint Series No.415, (1988), available on-line at:
`http://conservancy.umn.edu/bitstream/4838/1/415.pdf`