Title: **Second-order schemes for conservation laws with discontinuous flux modelling clarifier-thickener units**

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Starting from the enunciation of an eminently practical problem, the authors wade into the theoretical complexities of analyzing the stability and accuracy of numerical schemes solving a hyperbolic equation of the form [1] over a discontinuous domain and with discontinuous boundary conditions. This is another paper on a rather long list of works[1−7] the authors have been publishing since 2003 on the subject and one of its objectives is to clarify some questions about mathematical analysis.

$$u_t + b(u)_x = A(u)_{xx} \tag{1}$$

Depending on the complexity of the functions used for the source term, Eq 1 could be a deceptively simple equation, with a rich range of behaviour for its solutions. In the case of clarifier-thickener units the flux depends discontinuously on the spatial coordinate because the feed splits into two bulk flows going in opposing directions. It is in this context that the authors introduce TVD based second-order accurate finite difference schemes for the solution of the practical problem.

The major novelty of the present work is a scheme based on a new flux-total variation diminishing (FTVD) method although a simple TVD is also considered.

The authors described the error analysis and convergence analysis in great detail. They show the results of many relevant numerical experiments where it is shown that the proposed schemes perform significantly better than first order schemes.

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